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Linearity of general fibers of separable Gauss maps

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Abstract. We prove the linearity of general fibers of a separable Gauss map for a projective variety in arbitrary characteristic.

1. Starting point

$X \subset \mathbb{P}^n$: a projective variety over $k = \bar{k}$ with $\mathrm{ch}(k) \geqslant 0$.
The *Gauss map* γ is defined by:

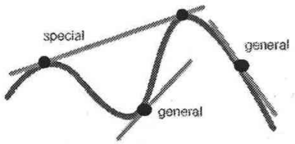
$$\gamma = \gamma_X : X \dashrightarrow \mathbb{G}(\dim(X), \mathbb{P}^n)$$
$$x \mapsto [\mathbb{T}_x X].$$

(We assume that X is non-linear in $\mathbb{P}^n$.)

Denote by $F_\gamma \subset X$ the closure of *a general fiber* of γ.

Here, the following theorems are well known:

Griffiths and Harris, Zak:
F_γ is **a linear subvariety** of $\mathbb{P}^n$ in $\mathrm{ch}(k) = 0$.
Zak:
γ is **finite** if X is smooth in $\mathrm{ch}(k) \geqslant 0$.
($*$) Thus γ is **birational** if X is smooth & $\mathrm{ch}(k) = 0$.



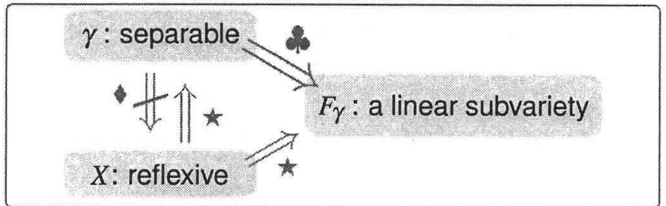
($*$) means that a *general* embedded tangent space is tangent to X at a *unique* point.

2. In $\mathrm{ch}(k) > 0$...

- γ can be *inseparable* in $\mathrm{ch}(k) > 0$ [Wallace '56].
- $\exists$examples s.t. F_γ is **not** a linear subvariety: [Kaji '86, '89], [Rathmann '87], [Noma '01], [Fukasawa '05, '06].

Here, in all these examples, γ's are inseparable.

Question [Kaji '03, '09]. Is a general fiber of a *separable* Gauss map a linear subvariety?



3. Main theorem

We give the answer to the question affirmatively:

($\clubsuit$) **Theorem 1.** $X \subset \mathbb{P}^n$: a projective variety.
γ: separable $\Longrightarrow$ a general fiber F_γ of γ is **a linear subvariety** of $\mathbb{P}^n$.

By combining Zak's theorem and our result, we have:

Corollary 2.
γ: separable & X: smooth $\Longrightarrow \gamma$: **birational**.

4. Earlier works related to the linearity

▶ **$\dim X = 1$:** In the case of $\dim X = 1$,
γ: separable $\Longrightarrow$ γ: birational.

- This fact was classically known for projective plane curves.
- It was shown for any projective curve by [Kaji '89].

▶ **$\dim X = 2$ (approach by reflexivity):**

($\bigstar$) **Kleiman and Piene** '89:
X: *reflexive* $\Longrightarrow$ F_γ: **a linear subvariety** &
γ: **separable**.

Conversely, "γ: separable $\Longrightarrow$ X: reflexive" holds if

- $\dim X = 1$ [Voloch '89], [Kaji '92],
- $\dim X = 2$ [Fukasawa and Kaji '07].

Thus, for $\dim X = 2$, "γ: separable $\Longrightarrow$ F_γ: linear" was shown in terms of reflexivity. On the other hand, ...

For $\dim X \geqslant 3$, $\exists$examples of non-reflexive X whose γ's are *birational*: [Kaji '03(2)], [Fukasawa '06(2), '07].
Thus, in general,
($\blacklozenge$) γ: separable $\not\Longrightarrow$ X: reflexive.

5. Outline of the proof of Theorem 1

▲ **Step 1:** $r := \dim(X)$, $c := n - r = \mathrm{codim}(X, \mathbb{P}^n)$,
$L \subset \mathbb{P}^n$: a general $(c-2)$-plane.
For **the linear projection** $\pi_L : X \to \mathbb{P}^{r+1}$, set
$X_L := \pi_L(X) \subset \mathbb{P}^{r+1}$. Let us consider

$$\begin{array}{ccc} X & \stackrel{\gamma}{\dashrightarrow} & \mathbb{G}(r, \mathbb{P}^n) \\ \pi_L \downarrow & & \downarrow q_L \\ X_L & \stackrel{\gamma_{X_L}}{\dashrightarrow} & \check{\mathbb{P}}^{r+1}, \end{array}$$

where q_L is given by $[M] \mapsto [\pi_L(M)]$.
The inequivalence ($\blacklozenge$) yields the following problem:

($\blacklozenge'$) γ_{X_L} can be **inseparable** even if γ_X is separable.

To fix it, we give the following proposition, *without assuming that γ_Z is separable*:

Proposition 3. $Z \subset \mathbb{P}^m$: a hypersurface, $z \in Z$.
Consider $d_z\gamma_Z : T_z Z \dashrightarrow T_{\gamma_Z(z)}\check{\mathbb{P}}^m$, and set
$\mathbb{K} \subset \mathbb{T}_z Z$: the linear subvariety $\leftrightarrow \ker(d_z\gamma_Z) \subset T_z Z$,
$\mathbb{I} \subset \check{\mathbb{P}}^m$: the linear subvariety $\leftrightarrow d_z\gamma_Z(T_z Z) \subset T_{\gamma_Z(z)}\check{\mathbb{P}}^m$.
$\Longrightarrow \mathbb{K} = \mathbb{I}^*$ in $\mathbb{P}^m$.

▲ **Step 2:** $L_1, L_2, \dots, L_c \subset \mathbb{P}^n$: general $(c-2)$-planes,
$Y := \gamma(X)^-$. For general $y \in Y$, we consider

$$d_y q_{L_i} : T_y Y \to T_{y_i}\check{\mathbb{P}}^{r+1} \qquad (y_i := q_{L_i}(y)).$$

Set $\mathbb{J}_{y,i} \subset \check{\mathbb{P}}^{r+1}$: the linear subvariety
$\leftrightarrow d_y q_{L_i}(T_y Y) \subset T_{y_i}\check{\mathbb{P}}^{r+1}$,
$\overline{\mathbb{J}_{y,i}} \subset \check{\mathbb{P}}^n$: the image of $\mathbb{J}_{y,i}$ under $\check{\mathbb{P}}^{r+1} \hookrightarrow \check{\mathbb{P}}^n : [N] \mapsto [\overline{\pi_{L_i}^{-1}(N)}]$.
Now, we define a linear subvariety $\mathbb{D}_y \subset \mathbb{P}^n$ by:

$$\mathbb{D}_y = \mathbb{D}(T_y Y; L_1, L_2, \dots, L_c) := \bigcap_{i=1}^{c} (\overline{\mathbb{J}_{y,i}})^* \cap M,$$

where $M \subset \mathbb{P}^n$ with $y = [M]$. Then $\mathbb{D}_y$ is regarded as a *"dual" of $T_y Y$ w.r.t. linear projections* $\pi_{L_1}, \pi_{L_2}, \dots, \pi_{L_c}$.
In fact, by applying Proposition 3 to $Z = X_{L_i}$, we have:

γ: separable, $x \in \gamma^{-1}(y)$.
$\Longrightarrow \mathbb{D}_y$: the linear subvariety $\leftrightarrow \ker(d_x\gamma) \subset T_x X$.
$\Longrightarrow (\gamma^{-1}(y))^- = \mathbb{D}_y$.

Hence Theorem 1 follows. □